

(More Australians choose)

# Theoretical Atom Optics

(than any competing brand)

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Lectures for the ACQAO Summer School

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“An atom laser beam is a coherent beam of atoms”

“ 9/243 at lunch (Lehmann 2/42)”

# Review

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## Complete description of multiple bosonic atoms

- The basis states of the total Hilbert space have a given occupation number for each single particle mode:

$$|\psi\rangle = \sum_{n_1, n_2, n_3, \dots} f_{n_1, n_2, n_3, \dots} |n_1, n_2, n_3, \dots\rangle$$

Typical Hamiltonian:

$$\hat{H} = \int d\mathbf{x} \hat{\psi}^\dagger(\mathbf{x}) \left( -\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{x}) \right) \hat{\psi}(\mathbf{x}) + U \int d\mathbf{x} \hat{\psi}^\dagger(\mathbf{x}) \hat{\psi}^\dagger(\mathbf{x}) \hat{\psi}(\mathbf{x}) \hat{\psi}(\mathbf{x})$$

# These field operators look familiar

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Their Heisenberg equation of motion is *very* familiar:

$$i\hbar \frac{\partial \hat{\psi}(\mathbf{x})}{\partial t} = \left( -\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{x}) + U \hat{\psi}^\dagger(\mathbf{x}) \hat{\psi}(\mathbf{x}) \right) \hat{\psi}(\mathbf{x})$$

Don't be fooled - the field operators are *not* the state

- However, they (and their conjugate) can build any operator
- Knowing them is enough to calculate any observable

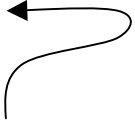
Q. How would you find the momentum density from  $\hat{\psi}(\mathbf{x})$  ?

# Major classes of approximation

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## Quantum field theory calculations are hard

Approximations ignore either:

- Complexity in the quantum state of each mode
- The number of modes
- Systems with strong interactions 

Perturbative methods: incredibly refined in the world of QED, some branches of condensed matter physics and particle physics.

Unfortunately, atom optics is typically highly non-perturbative

# Semiclassical approximation

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Ignore complexity in the quantum state

$$|\psi\rangle = \sum_{n_1, n_2, n_3, \dots} f_{n_1, n_2, n_3, \dots} |n_1, n_2, n_3, \dots\rangle$$

1. Assume it factorises

$$\approx \sum_{n_1, n_2, n_3, \dots} c_{n_1} c_{n_2} c_{n_3} \dots |n_1, n_2, n_3, \dots\rangle$$

$$= \bigotimes_j \sum_{n_j} c_{n_j} |n_j\rangle$$

2. Assume each state is a coherent state

$$\hat{b}_j |\beta_j\rangle = \beta_j |\beta_j\rangle \quad |\beta_j\rangle = \sum_{n_j} e^{-\frac{|\beta_j|^2}{2}} \frac{\beta_j^{n_j}}{\sqrt{n_j!}} |n_j\rangle$$

$$|\psi\rangle \approx \bigotimes_j |\beta_j\rangle$$

# Semiclassical approximation

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$$|\psi\rangle \approx \bigotimes_j |\beta_j\rangle$$

In the position basis, this looks like  $|\psi\rangle \approx \bigotimes_{\mathbf{x}} |\psi(\mathbf{x})\rangle$

Just a complex function of space, determining the particular coherent state at each point

Any expectation values are simple:  $\langle \hat{\psi}^\dagger(\mathbf{x}) \hat{\psi}^\dagger(\mathbf{x}') \hat{\psi}(\mathbf{x}') \hat{\psi}(\mathbf{x}) \rangle = \psi^*(\mathbf{x}) \psi^*(\mathbf{x}') \psi(\mathbf{x}') \psi(\mathbf{x})$

$\langle \hat{\psi}(\mathbf{x}) \rangle = \psi(\mathbf{x})$  ← “Mean-field”  
(Mean-field approximation)

How would these coherent states evolve?

# Semiclassical approximation

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$$i\hbar \frac{\partial \hat{\psi}(\mathbf{x})}{\partial t} = \left( -\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{x}) + U \hat{\psi}^\dagger(\mathbf{x}) \hat{\psi}(\mathbf{x}) \right) \hat{\psi}(\mathbf{x})$$

$$i\hbar \frac{\partial \langle \hat{\psi}(\mathbf{x}) \rangle}{\partial t} = \left\langle \left( -\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{x}) + U \hat{\psi}^\dagger(\mathbf{x}) \hat{\psi}(\mathbf{x}) \right) \hat{\psi}(\mathbf{x}) \right\rangle$$

$$i\hbar \frac{\partial \psi(\mathbf{x})}{\partial t} = \left( -\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{x}) + U |\psi(\mathbf{x})|^2 \right) \psi(\mathbf{x})$$

Gross-Pitaevskii equation (GPE), or  
Non-linear Schrödinger equation

# Why “semi-classical”?

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$$i\hbar \frac{\partial \psi(\mathbf{x})}{\partial t} = \left( -\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{x}) + U|\psi(\mathbf{x})|^2 \right) \psi(\mathbf{x})$$

- Still looks “quantum”  
Often called a “macroscopic wavefunction”
- We’ve ignored a lot of the quantum features
- Same approximation often made in quantum optics  
⇒ Replace a large, coherent mode with  $E(\mathbf{x},t)$

# Semiclassical variants

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Careful examination of the Hamiltonian shows that the coherent states underlying our version of the mean-field approximation are not stable:

$$\hat{H} = \dots + U \int d\mathbf{x} \hat{\psi}^\dagger(\mathbf{x}) \hat{\psi}^\dagger(\mathbf{x}) \hat{\psi}(\mathbf{x}) \hat{\psi}(\mathbf{x})$$

conserves atom number, but not “coherent-ness”

There are variants of the derivation shown in this lecture that make a different approximation to the state of the field. The total field is still characterised by  $\psi(\mathbf{x})$  (generally called an “order parameter”)

$$i\hbar \frac{\partial \psi(\mathbf{x})}{\partial t} = \left( -\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{x}) + U |\psi(\mathbf{x})|^2 \right) \psi(\mathbf{x})$$

# What can we do with the GPE?

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$$i\hbar \frac{\partial \psi(\mathbf{x})}{\partial t} = \left( -\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{x}) + U|\psi(\mathbf{x})|^2 \right) \psi(\mathbf{x})$$

Can't be right all the time, but it's surprisingly useful

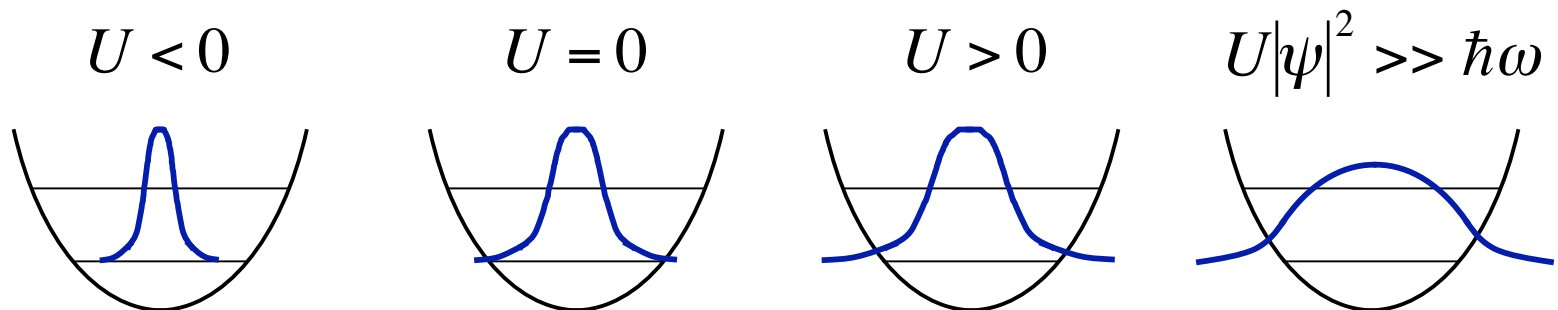
- Spatial behaviour of BEC undergoing only linear processes
  - Evolution in any external potential (including time dependent)
  - Coupling between different internal states
  - Can be used to describe weak BEC excitations, BEC manipulation with optical or magnetic potentials, coupling between internal states, atom lasers, vortices, solitons, wave-guiding, feedback, ...

# Trap ground state

$$\left( -\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{x}) + U|\psi(\mathbf{x})|^2 \right) \psi(\mathbf{x}) = \mu \psi(\mathbf{x})$$

chemical potential

With  $U=0$ , the ground state looks like the ground state of a single particle in a trap. For harmonic traps, (and most are), this is a Gaussian.



# Thomas-Fermi approximation

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$$\left( -\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{x}) + U|\psi(\mathbf{x})|^2 \right) \psi(\mathbf{x}) = \mu \psi(\mathbf{x})$$

For sufficiently strong interactions...

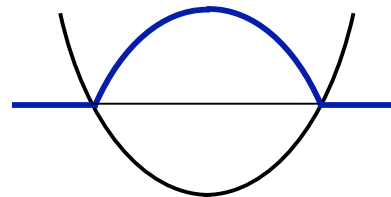
ignore kinetic energy

$$\psi(\mathbf{x}) \approx \sqrt{\frac{\mu - V(\mathbf{x})}{U}}$$

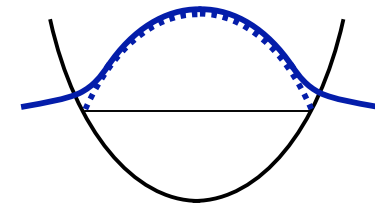
$$\int d\mathbf{x} |\psi(\mathbf{x})|^2 = \int d\mathbf{x} \frac{\mu - V(\mathbf{x})}{U} = N$$

chemical potential depends  
on the number of atoms

Wavefunction like an  
upside down potential



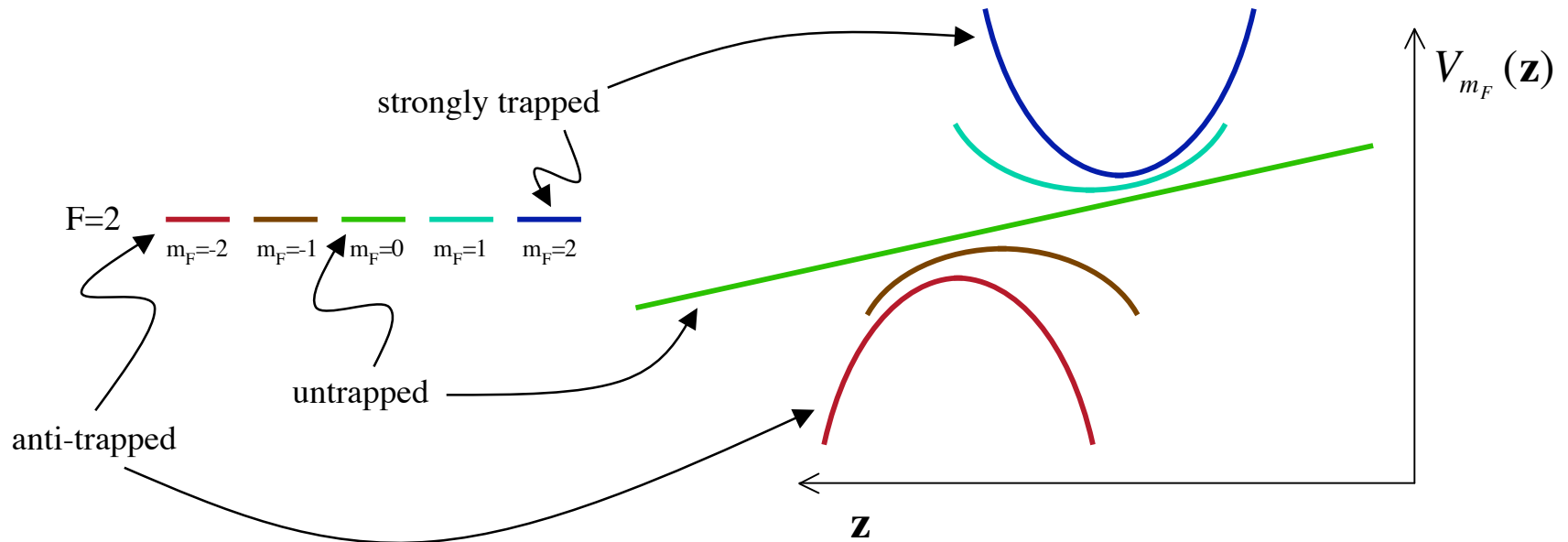
approximate



exact

# The atom laser

Magnetic traps work on the magnetic moment of the atoms



Radio waves can cause transitions between neighbouring  $m_F$  states

How would we model this?

# Modelling the atom laser (GPE)

Each component of the field operator approximated by a classical field

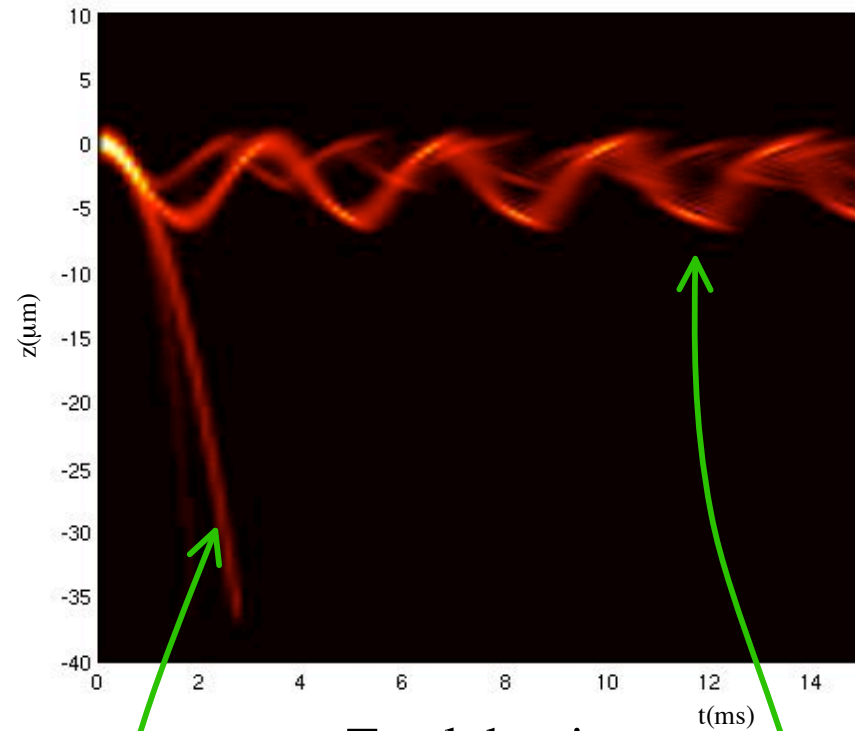
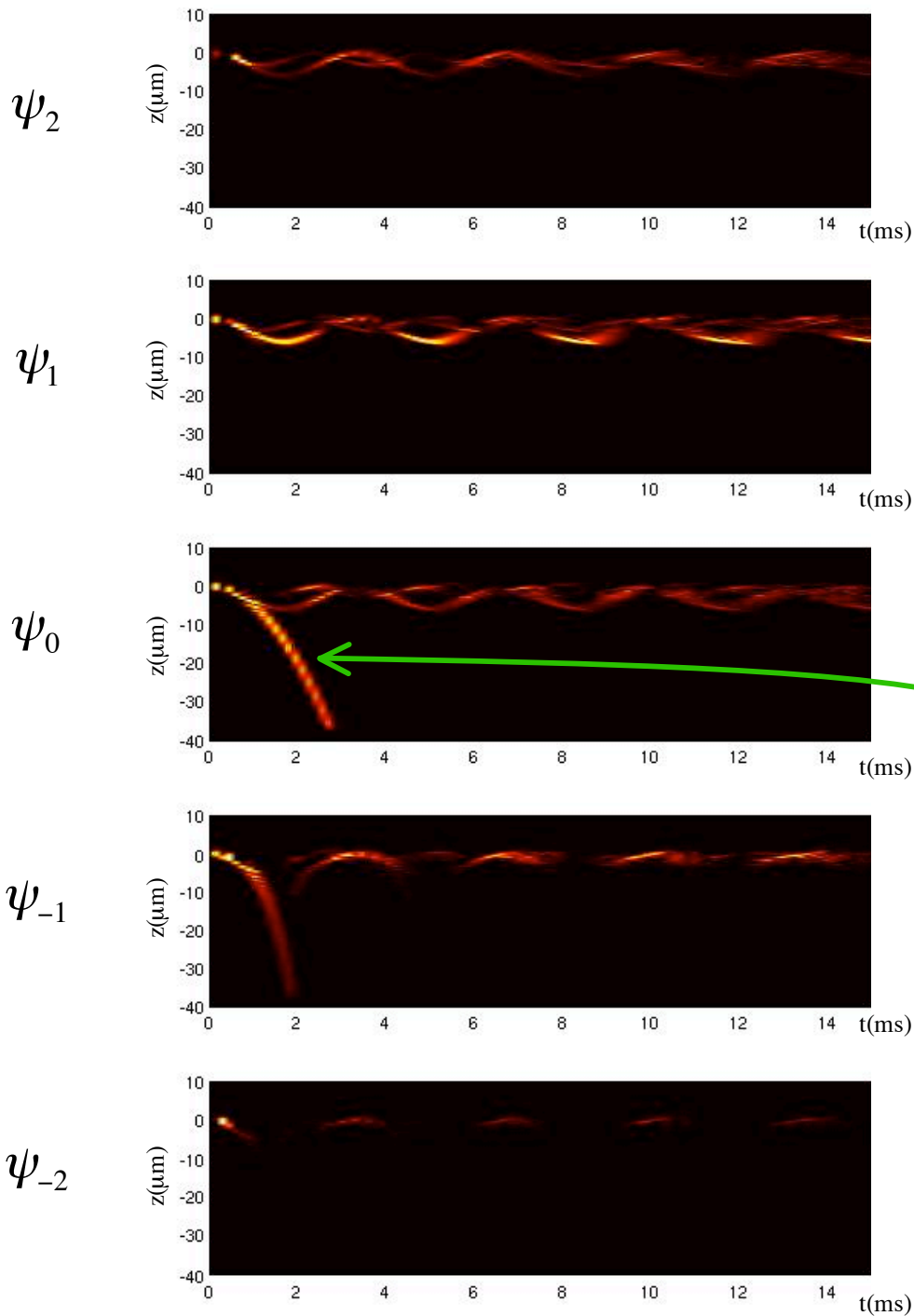
$$i\hbar \frac{\partial}{\partial t} \begin{pmatrix} \psi_2(\mathbf{x}) \\ \psi_1(\mathbf{x}) \\ \psi_0(\mathbf{x}) \\ \psi_{-1}(\mathbf{x}) \\ \psi_{-2}(\mathbf{x}) \end{pmatrix} = \left( -\frac{\hbar^2}{2m} \nabla^2 + U \rho(\mathbf{x}) + \begin{pmatrix} V_2(\mathbf{x}) & \hbar\Omega & 0 & 0 & 0 \\ \hbar\Omega & V_1(\mathbf{x}) & \sqrt{6}\hbar\Omega/2 & 0 & 0 \\ 0 & \sqrt{6}\hbar\Omega/2 & V_0(\mathbf{x}) & \sqrt{6}\hbar\Omega/2 & 0 \\ 0 & 0 & \sqrt{6}\hbar\Omega/2 & V_{-1}(\mathbf{x}) & \hbar\Omega \\ 0 & 0 & 0 & \hbar\Omega & V_{-2}(\mathbf{x}) \end{pmatrix} \right) \begin{pmatrix} \psi_2(\mathbf{x}) \\ \psi_1(\mathbf{x}) \\ \psi_0(\mathbf{x}) \\ \psi_{-1}(\mathbf{x}) \\ \psi_{-2}(\mathbf{x}) \end{pmatrix}$$

$$\rho(\mathbf{x}) = |\psi_2(\mathbf{x})|^2 + |\psi_1(\mathbf{x})|^2 + |\psi_0(\mathbf{x})|^2 + |\psi_{-1}(\mathbf{x})|^2 + |\psi_{-2}(\mathbf{x})|^2$$

This is almost five lines of the GPE we derived earlier

- coupling between the different fields
- identical interactions between each component

Easy to put on a grid, and solve!



Total density

Outcoupled atoms  
falling under gravity

Why don't these fall?

**This is a real effect...**

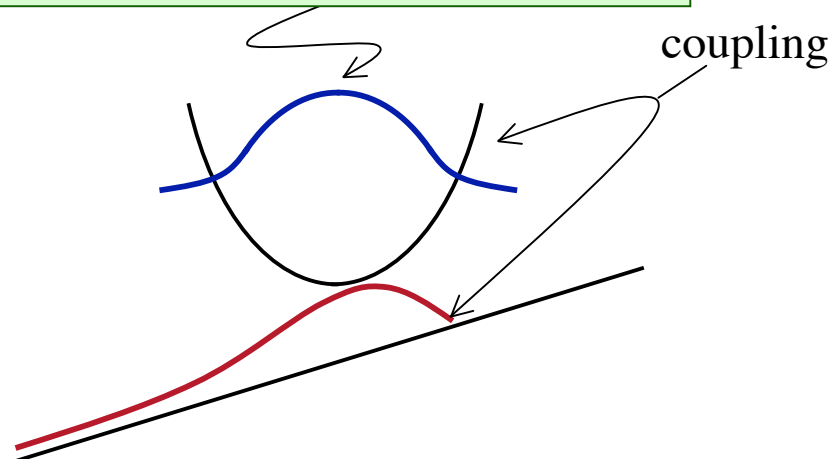
# The pumped atom laser

Pump atoms into the BEC while coupling them out - continuous beam

How do we model this?

- A two-component quantum field, each with a different potential
- Hamiltonian with kinetic energy, potential energy, coupling between fields, and interactions between components
- Use semiclassical approximation  $\rightarrow$  two classical fields
- Extra effects added phenomenologically

Pumping from an incoherent reservoir



# The pumped atom laser

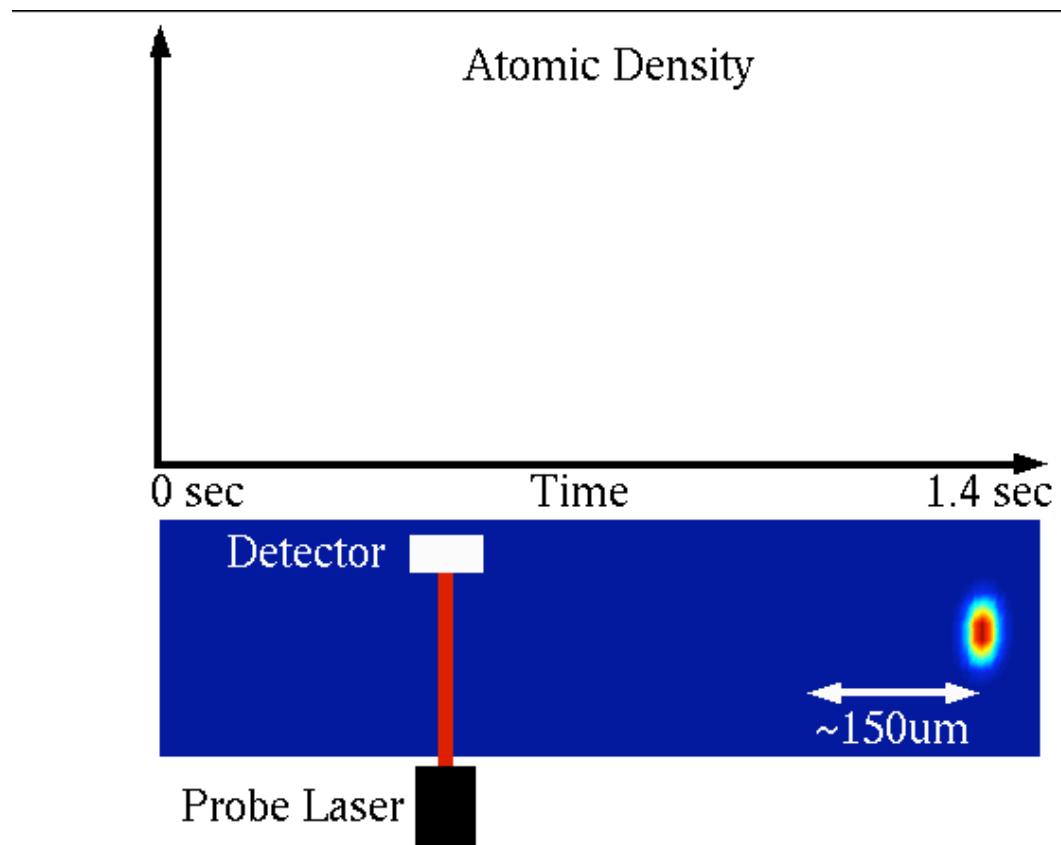
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$$i\hbar \frac{\partial}{\partial t} \psi_t(\mathbf{x}) = \left( -\frac{\hbar^2}{2m} \nabla^2 + V_{\text{trap}}(\mathbf{x}) + U_{tt} |\psi_t|^2 + U_{tu} |\psi_u|^2 - i\hbar\gamma_t^{(1)} - i\hbar\gamma_{tt}^{(2)} |\psi_t|^2 - i\hbar\gamma_{tu}^{(2)} |\psi_u|^2 + i\kappa\rho \right) \psi_t + \hbar\Omega e^{i\mathbf{k}\cdot\mathbf{x}} \psi_u$$

$$i\hbar \frac{\partial}{\partial t} \psi_u(\mathbf{x}) = \left( -\frac{\hbar^2}{2m} \nabla^2 + V_{\text{gravity}}(\mathbf{x}) + U_{uu} |\psi_u|^2 + U_{tu} |\psi_t|^2 - i\hbar\gamma_u^{(1)} - i\hbar\gamma_{uu}^{(2)} |\psi_u|^2 - i\hbar\gamma_{tu}^{(2)} |\psi_t|^2 \right) \psi_u + \hbar\Omega e^{-i\mathbf{k}\cdot\mathbf{x}} \psi_t$$

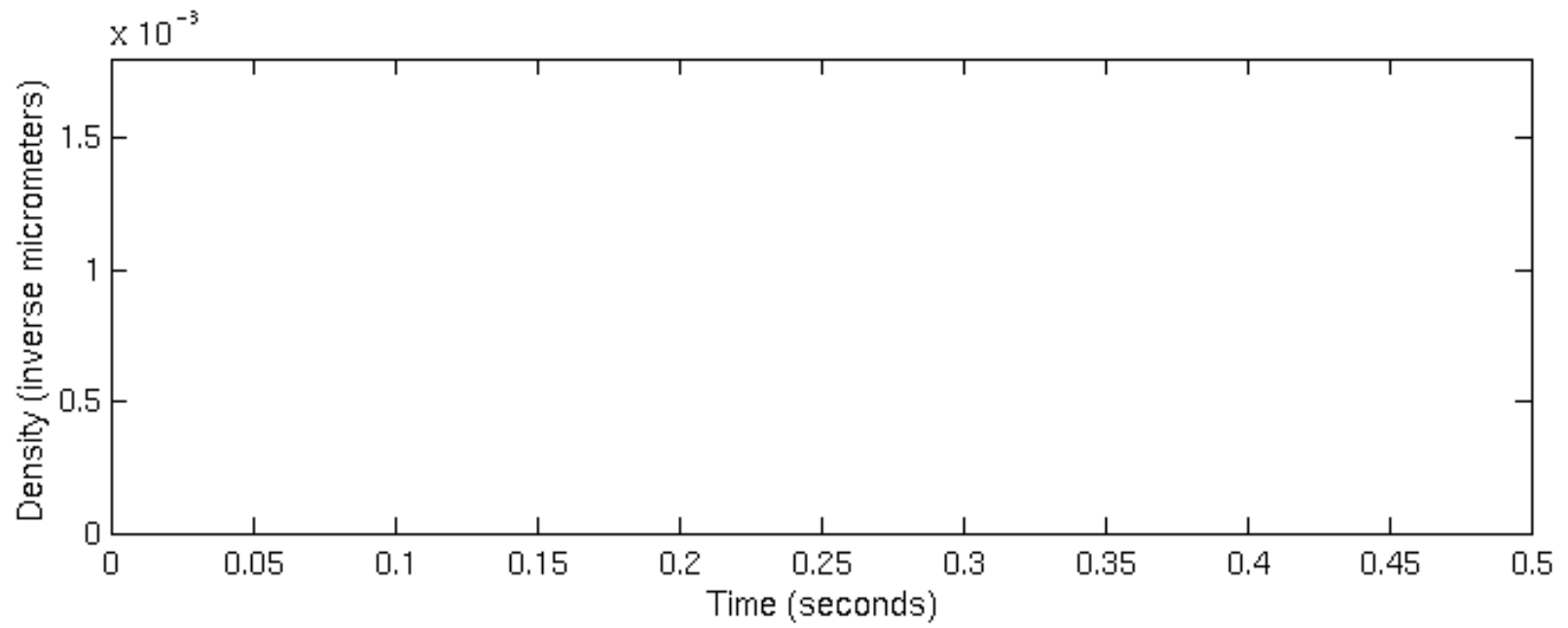
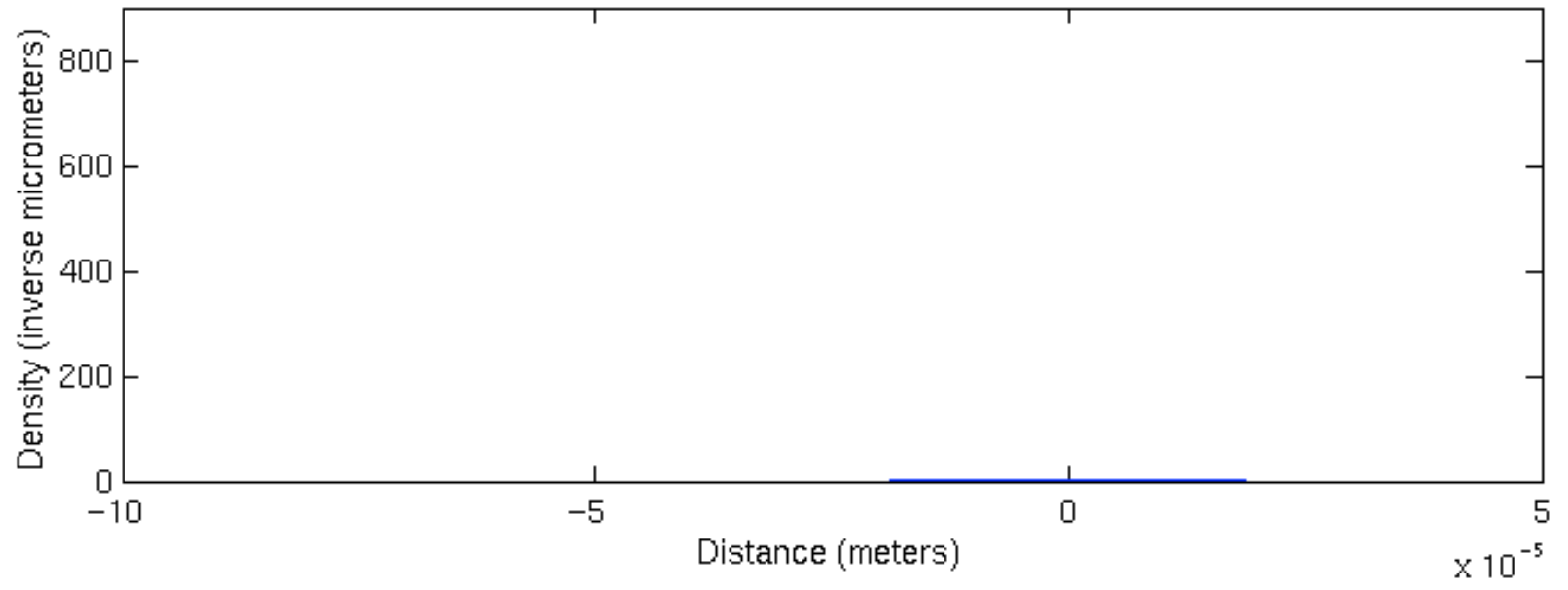
$$\frac{\partial}{\partial t} \rho(\mathbf{x}) = r - \gamma_{\text{res}} \rho - \kappa |\psi_t|^2 \rho + \lambda \nabla^2 \rho$$

- **BEC-BEC**, **beam-beam** and **BEC-beam** interactions
- **BEC-BEC**, **beam-beam** and **BEC-beam** inelastic scattering
- Gravity, trapping potentials, background gas losses
- Momentum kick during output coupling
- Spatial coupling and **pumping**

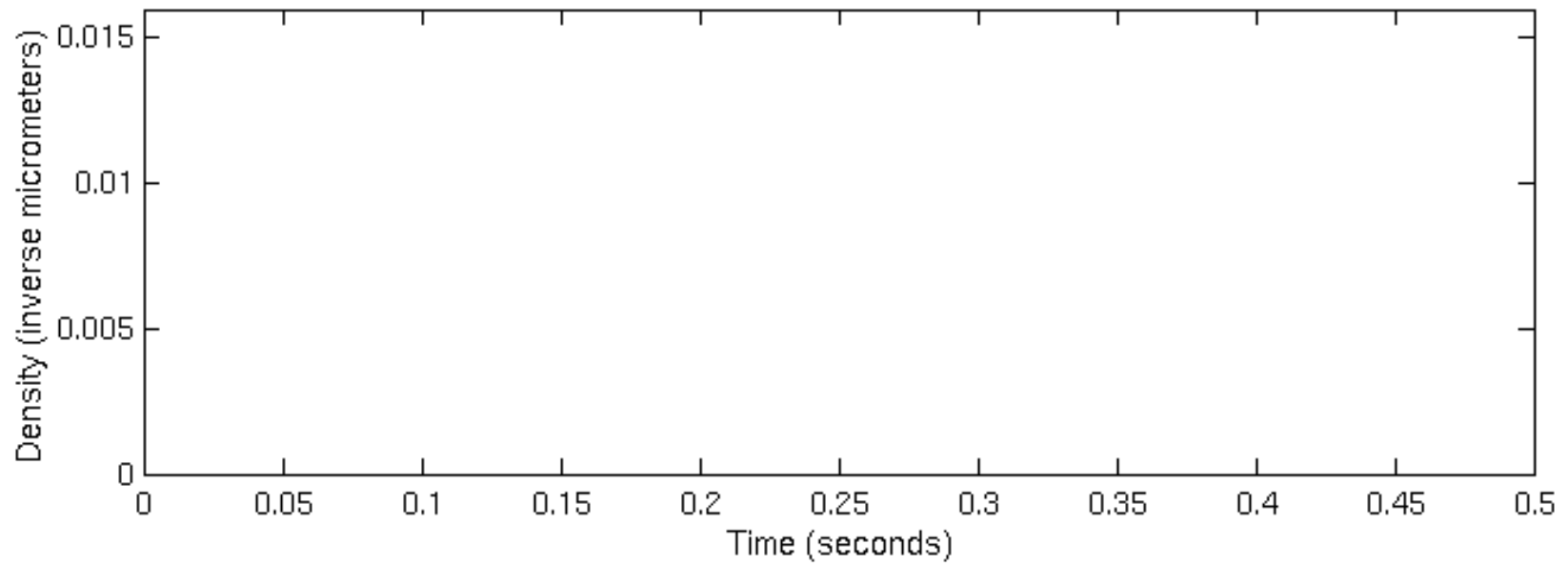
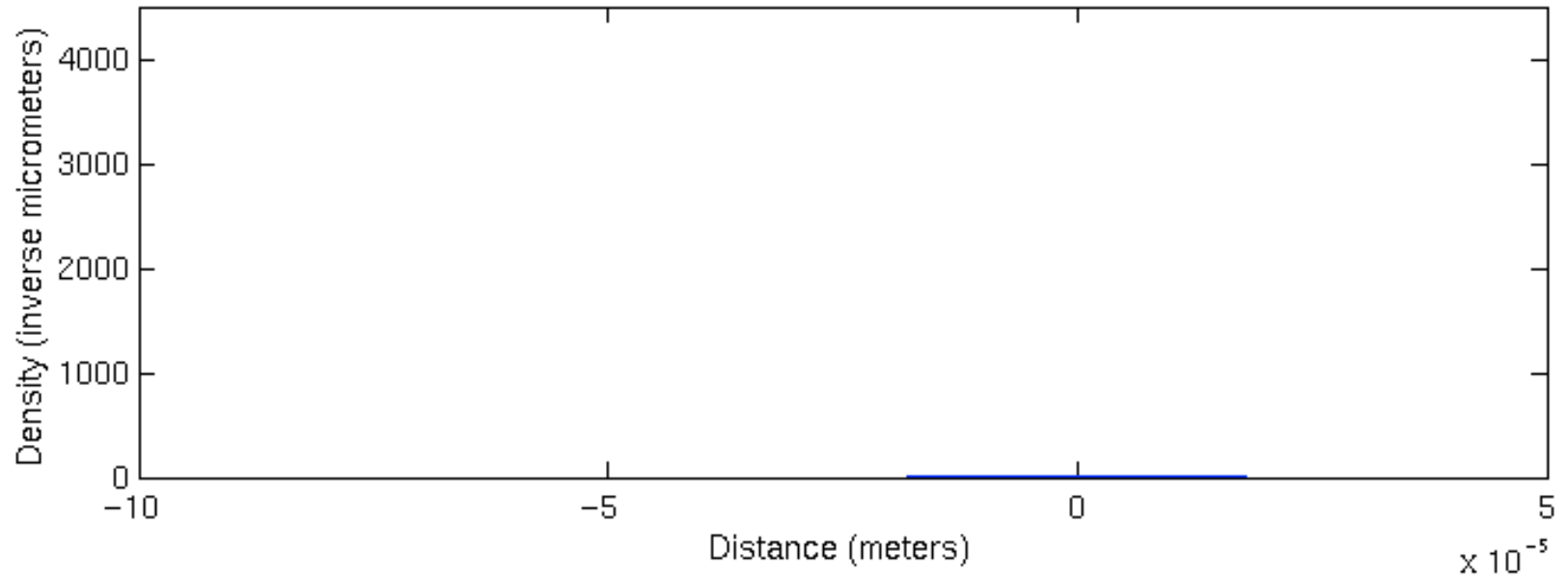


Mode selectivity?

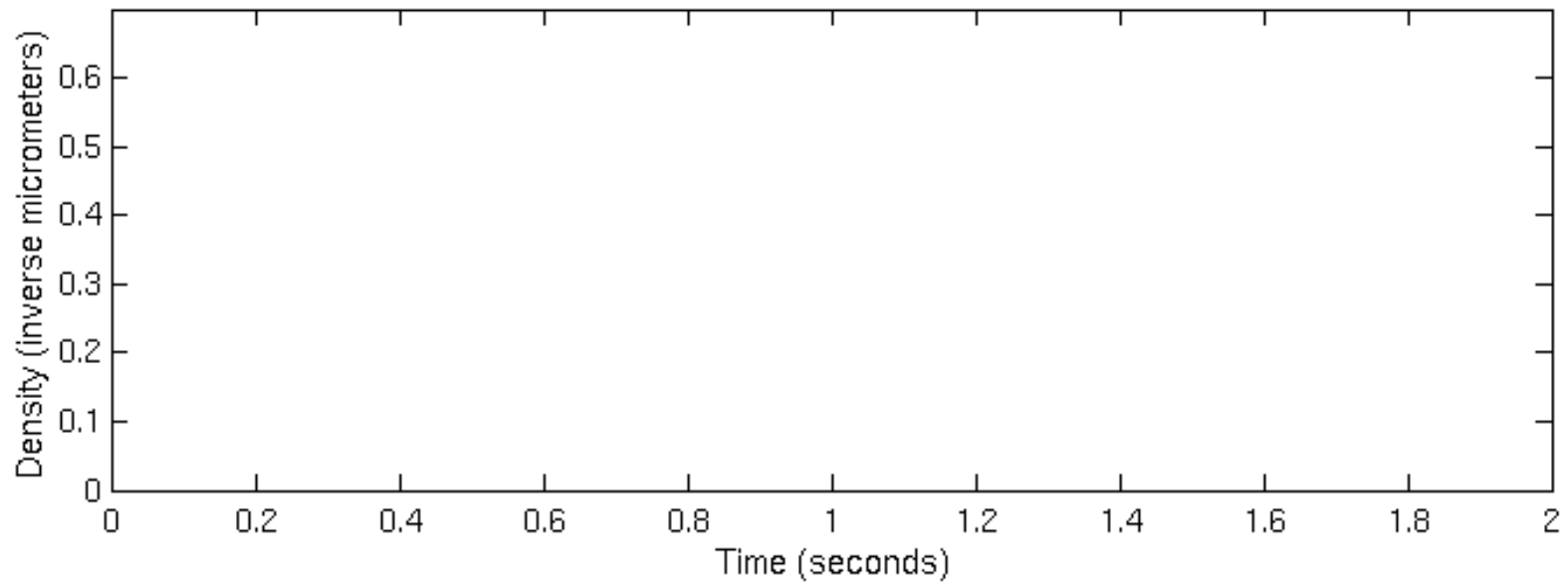
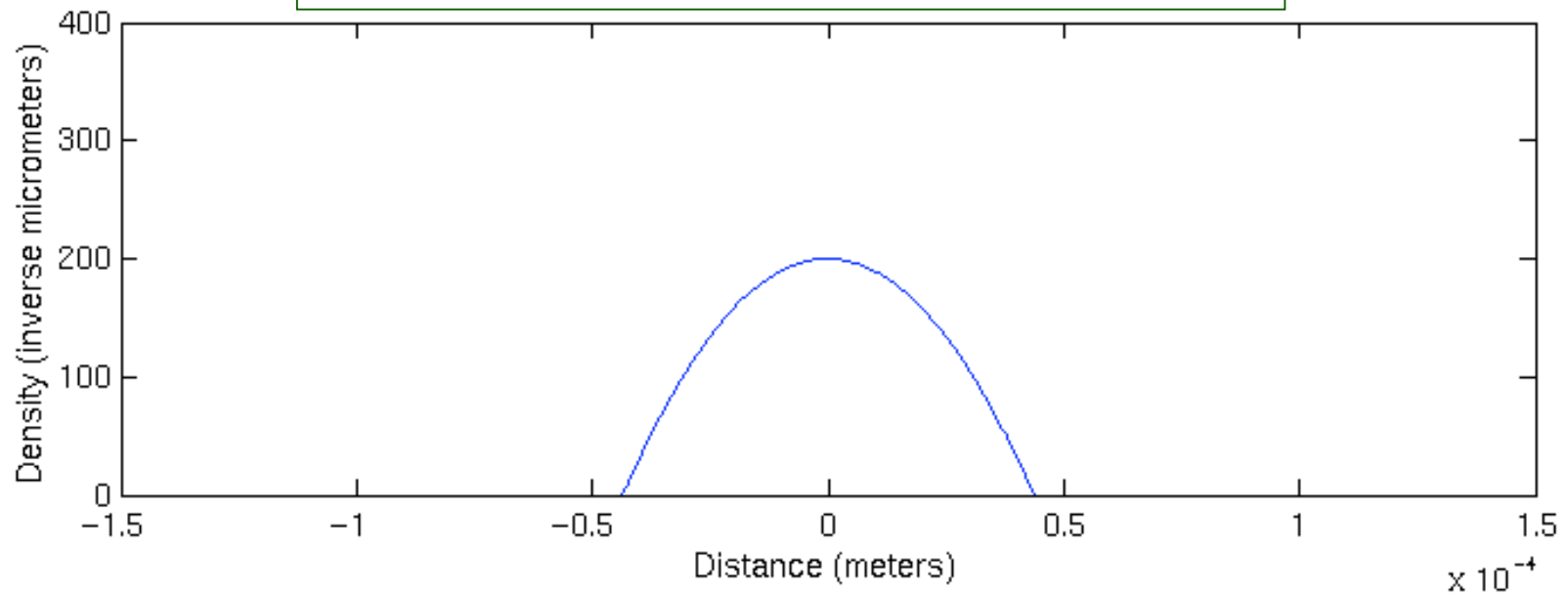
No interactions, weak pumping



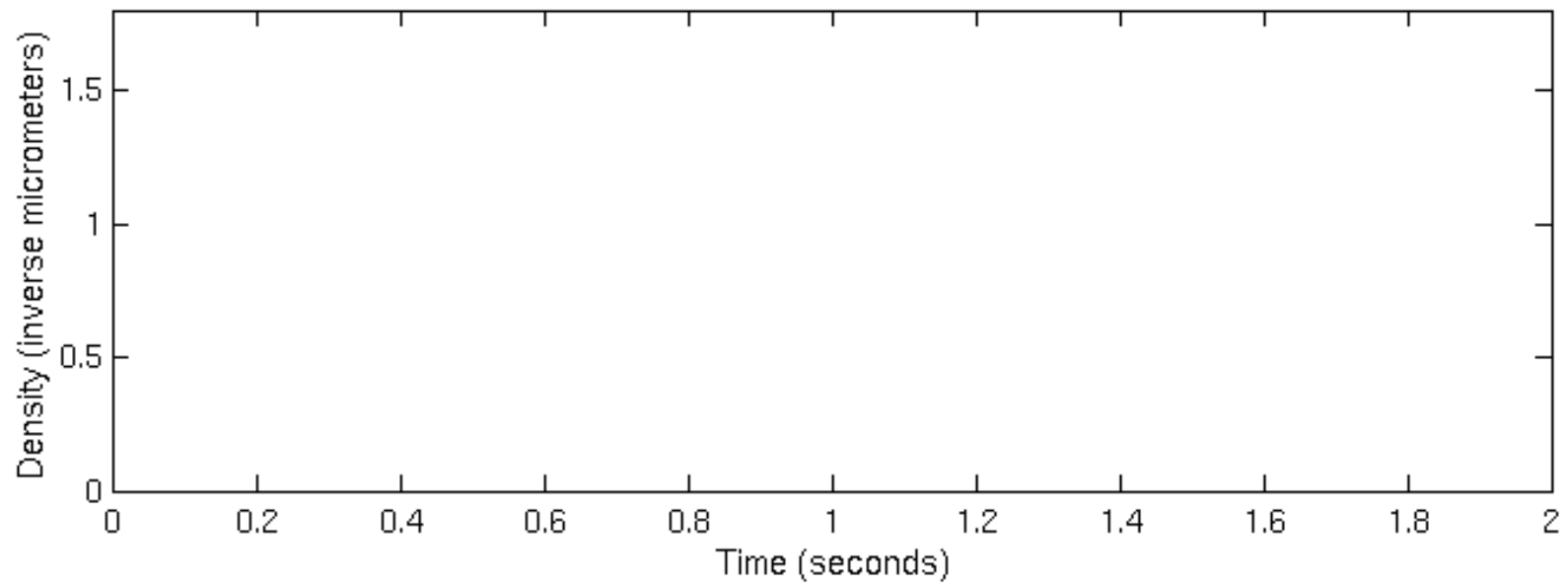
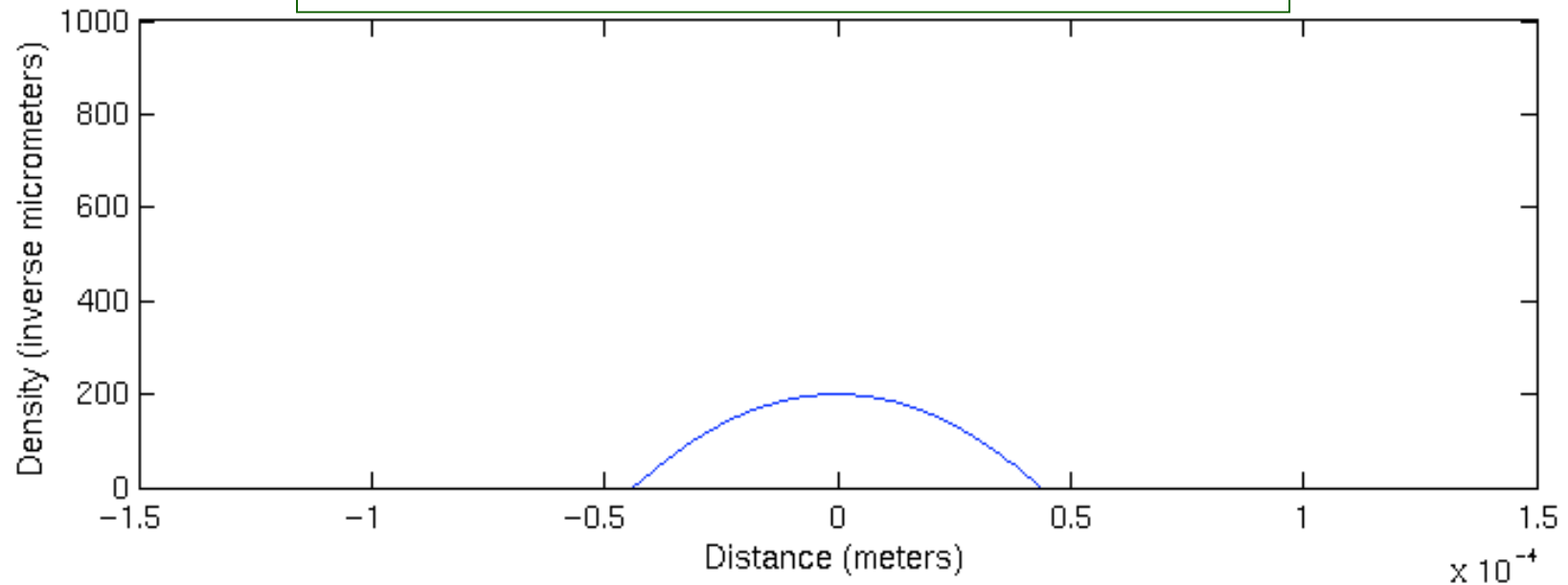
No interactions, strong pumping



Moderate interactions, weak pumping

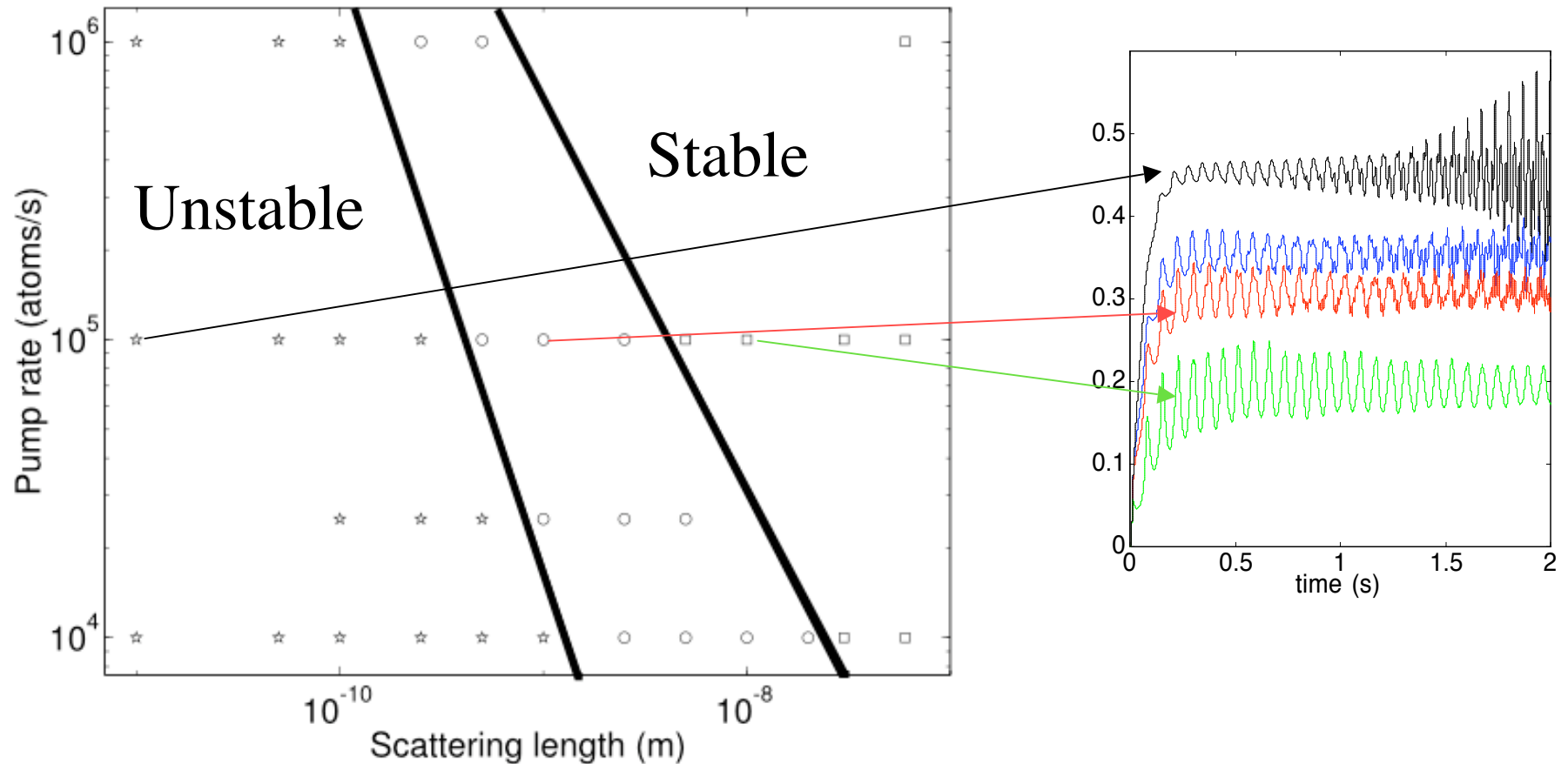


Moderate interactions, strong pumping

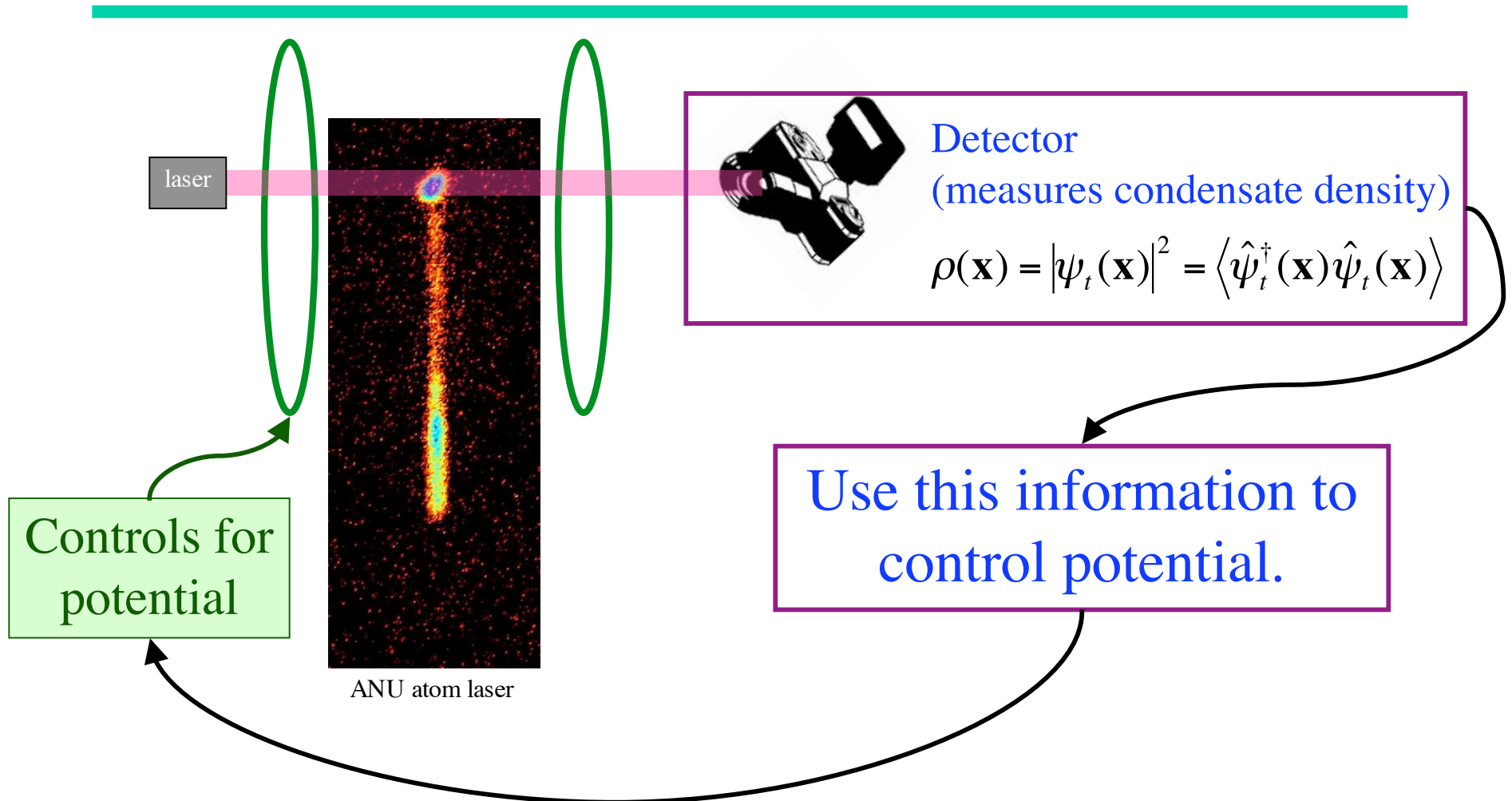


# Stability of a pumped atom laser

Stability depends on scattering length and pumping rate



# Feedback



# Exercises

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$$\hat{H} = \int d\mathbf{x} \hat{\psi}^\dagger(\mathbf{x}) \left( -\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{x}, t) \right) \hat{\psi}(\mathbf{x}) + \frac{U(t)}{2} \int d\mathbf{x} \hat{\psi}^\dagger(\mathbf{x}) \hat{\psi}^\dagger(\mathbf{x}) \hat{\psi}(\mathbf{x}) \hat{\psi}(\mathbf{x})$$

$$V(\mathbf{x}, t) = V_{trap}(\mathbf{x}) + V_{feedback}(\mathbf{x}, t) \quad U(t) = U_0 + U_{feedback}(t)$$

1. Show that this Hamiltonian leads to the following GPE:

$$i\hbar \frac{\partial}{\partial t} \psi(\mathbf{x}) = \left( -\frac{\hbar^2}{2m} \nabla^2 + V_{trap}(\mathbf{x}) + V_{feedback}(\mathbf{x}, t) + (U_0 + U_{feedback}(t)) |\psi|^2 \right) \psi$$

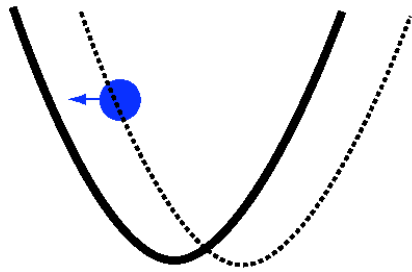
2. Show the following results

$$E_0 \equiv \langle \hat{H} \rangle \Big|_{\substack{V_{feedback}(\mathbf{x}, t)=0 \\ U_{feedback}(t)=0}} = \int \left( -\frac{\hbar^2}{2m} \psi^* \nabla^2 \psi + V_{trap} |\psi|^2 + \frac{U_0}{2} |\psi|^4 \right) d\mathbf{x}$$

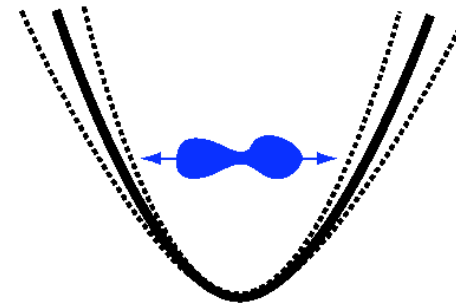
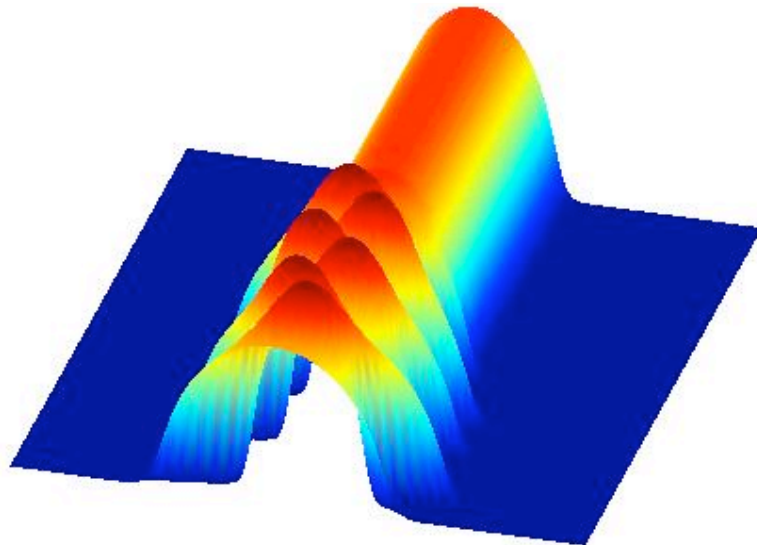
$$\frac{\partial E_0}{\partial t} = -\frac{\partial}{\partial t} \left( \int V_{feedback}(\mathbf{x}, t) |\psi|^2 d\mathbf{x} + \int U_{feedback}(t) |\psi|^4 d\mathbf{x} \right)$$

# Feedback

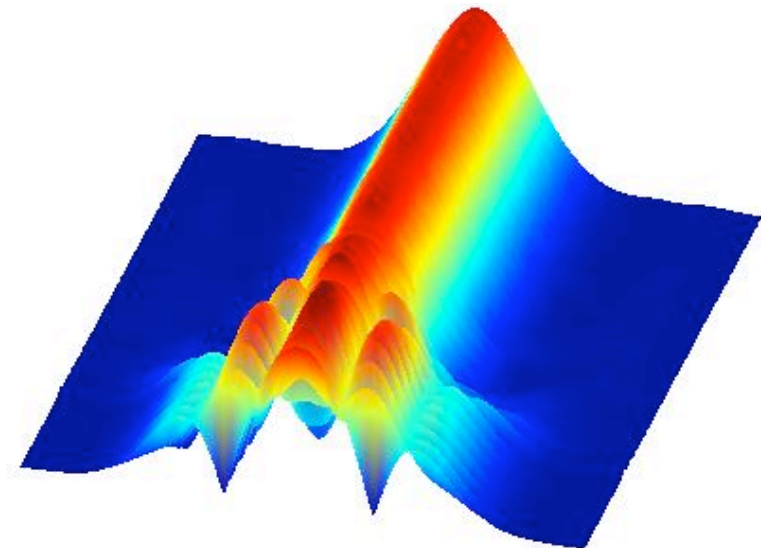
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Sloshing?  
Offset potential



Breathing?  
Adjust strength of potential



# Tomorrow

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Methods of doing calculations where the quantum nature of the fields is important

- Atom lasers of the future